Comparing results for a global metric from analytical perturbation theory and a numerical code

J. E. Cuchí, A. Molina, E. Ruiz

Abstract We compare the results obtained from analytical perturbation theory and the AKM numerical code for an axistationary spacetime built from matching a rotating perfect fluid interior with the equation of state $\varepsilon - 3p = 4B$ of the simple MIT bag model and an asymptotically flat exterior. We discuss the behaviour of the error in the metric components of the analytical approximation going to higher orders. Additionally, we check and comment the errors in multipole moments, central pressure and some other physical properties of the spacetime.

1 Introduction

The lack of stellar models in General Relativity –i. e. a stationary and axisymmetric perfect fluid interior matched with an asymptotically flat vacuum exterior– is in direct contrast with the importance they could have for the astrophysics of compact stars and in particular the determination of their possible compositions. One of such possibilities is the interesting case of strange matter. A versatile model for strange matter is a simple MIT bag model with equation of state (EOS) $\varepsilon - 3p = 4B$. We will study this stellar model with the results provided by the CMMR post-Minkowskian and slow rotation approximation scheme [3] and its behaviour when we go to higher orders of approximation. Also, we will give the relative error in these functions and quantities when compared with very precise numerical results obtained with the AKM code [1, 2].

J. E. Cuchí
Departamento de Física Fundamental, Universidad de Salamanca e-mail: jecuchi@gmail.com

A. Molina
Departamento de Física Fonamental, Universidad Autónoma de Barcelona e-mail: alfred.molina@ub.edu

E. Ruiz
Departamento de Física Fundamental, Universidad de Salamanca e-mail: eruiz@usal.es

2 The analytical and numerical metrics

The stationary and axisymmetric spacetime we study is the following. The interior $\mathscr{V}^-$ is filled with a perfect fluid with the EOS $\varepsilon - 3p = \varepsilon_0$ and in ridig rotation so that, if $\boldsymbol{\xi}, \boldsymbol{\chi}$ are the Killing vectors, its velocity is $\boldsymbol{u} = \psi(\boldsymbol{\xi} + \omega\boldsymbol{\chi})$, with ψ a normalization factor and ω a constant correspondig to the rotation speed of the fluid as seen by a distant observer. The exterior $\mathscr{V}^+$ is asymptotically flat vacuum and is matched with the interior imposing continuity of the metrics and their first derivatives on the $p = 0$ surface.

In CMMR, we solve the Einstein equations using a truncated multipolar post-Minkowskian approximation in spherical-like coordinates associated to harmonic ones. The post-Minkowskian parameter is $\lambda = m/r_s$, whith m the Newtonian mass of the source and r_s the coordinate radius of the static fluid; a different parameter $\Omega^2 = \omega^2 r_s^3/m$ (the ratio between Newtonian centrifugal and gravitational forces), gives the truncation point of the expansion in spherical harmonics, in this case preserving terms up to $\mathscr{O}(\Omega^3)$. The metric in each spacetime $\boldsymbol{g}^{\pm}(\lambda, \Omega)$ is decomposed in Minkowski $\boldsymbol{\eta}$ plus the deviation $\boldsymbol{h}^{\pm}(\lambda, \Omega)$ and then Einstein's equations can be solved iteratively. The spacetimes are matched on the $p = 0$ surface which can be expanded as $r_\Sigma = r_s\left[1 + \sigma\Omega^2 P_2(\cos\theta)\right] + \mathscr{O}(\Omega^4)$ with σ constant. Finally, the global metric depends only on ε_0, ω and r_s. In [4] (CGMR) we obtained the $\mathscr{O}(\lambda^{5/2}, \Omega^3)$ metric for the EOS $\varepsilon + (1-k)p = \varepsilon_0$. Here we use its ($k = 4$, $\varepsilon_0 = 4B$) subcase corresponding to the simple MIT bag model but now including terms up to $\mathscr{O}(\lambda^{9/2}, \Omega^3)$.

AKM is a multi-domain spectral code that gives the matched metric on a grid over a quadrant of finite size. The grid coordinates are $\{\rho, \zeta\}$, cylindrical associated to quasi-isotropic coordinates and the resolution is customizable. It also gives some physical properties, like the first mass and angular momentum multipole moments M_0 and J_1, baryon rest mass $M_{\rm b}$, circumpherential radius $R_{\rm circ}$, binding energy $E_{\rm b}$, polar and equatorial coordinate radii $r_{\rm p}$, $r_{\rm e}$, polar redshift $z_{\rm p}$ and central pressure and specific enthalpy $p_{\rm c}$, $h_{\rm c}$. To build a stellar model, it needs goal values for any

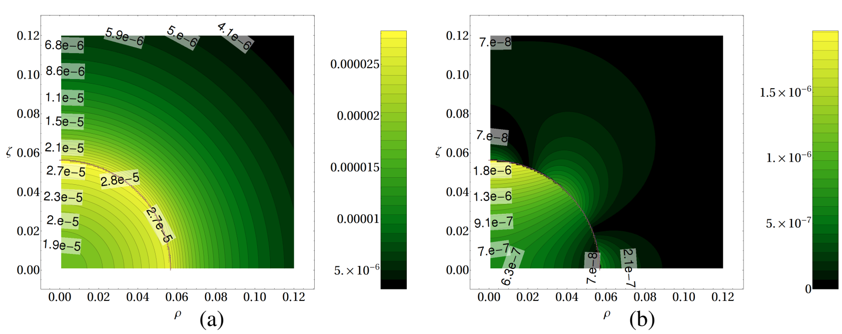

Fig. 1 Relative error for constant density in g_{tt} using $M_0 = 8 \times 10^{-4}$, $\omega = 0.2$ using CMMR up to: (a) $\mathscr{O}(\lambda^{5/2}, \Omega^3)$; (b) $\mathscr{O}(\lambda^{9/2}, \Omega^3)$. The thin dotted lines represent the AKM and CMMR surfaces (indistinguishable in this picture size)

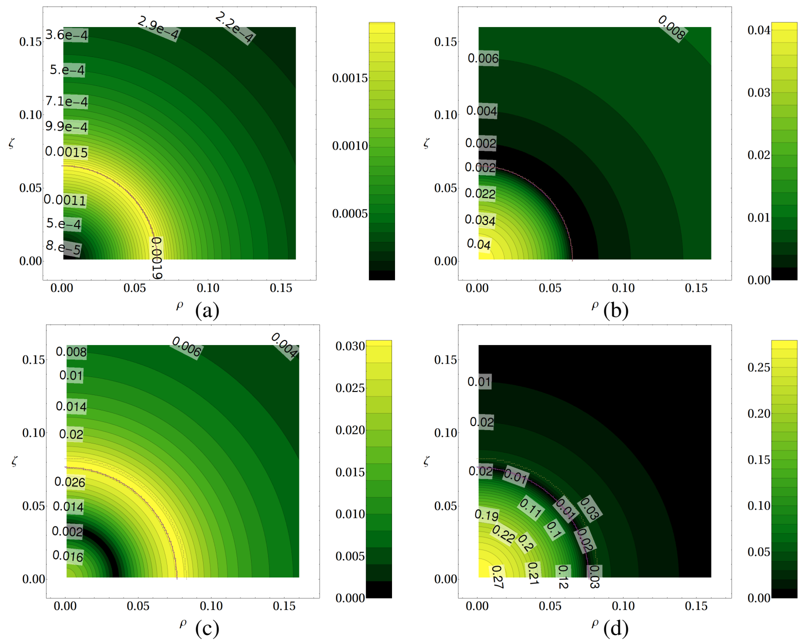

Fig. 2 Relative error between $\mathcal{O}(\lambda^{9/2}, \Omega^3)$ CMMR and $n = 20$ AKM in (a-b) g_{tt} and $g_{t\varphi}$ for $M_0 = 8\times 10^{-3}$, $\omega = 0.24$ ($\lambda \approx 0.077$, $\Omega \approx 0.059$); (c-d) g_{tt} and $g_{t\varphi}$ for $M_0 = 0.0184$, $\omega = 0.24$ ($\lambda \approx 0.12$, $\Omega \approx 0.059$). The thin dotted lines represent the AKM and CMMR surfaces.

pair of these quantities as well as to fix some parameters to specify the EOS and provide an initial data file. The code is able to obtain initial data for different EOS if one manages to avoid unphysical configurations in the process. The precision it gets depends on the number n of Chebyshev polynomials used, and can reach machine accuracy for high enough n when the deformation of the source is not extreme. We fix it here to $n = 20$.

3 Comparison and results

We build stellar models for different values of (M_b, ω) and compare the metrics on the cylindrical-like coordinate grid of AKM. Working in units where $(c = G = B = 1)$, the $k = 4$ CGMR has only (r_s, ω) as free parameters. Unlike ω, r_s has no equivalent in AKM, so we must adjust its value. Hence, for each model we take the value of one of the AKM physical quantities and equating it to its CMMR counterpart we get r_s. Using M_0, p_c, and J_1 for this gives similar results. Here we use M_0.

The results of Fig. 2 show the relative error in the metric $(g_{tt}, g_{t\phi})$ components for $M_0 = 0.51\,M_\odot$ and the typical $M_0 = 1.39\,M_\odot$ for a moderately fast rotation frequency $\nu = 102$ Hz near the source. The first case have a reasonable error (higher in $g_{t\phi}$)

and the second must be improved. Going to higher orders in the λ approximation is now automatic with the help of our *Mathematica* subroutines, but improving the slow-rotation approximation is more cumbersome. Nevertheless, Fig. 1 shows the improvement for a $\varepsilon = \varepsilon_0$ EOS of moving from $\mathscr{O}(\lambda^{5/2}, \Omega^3)$ to $\mathscr{O}(\lambda^{9/2}, \Omega^3)$. It make us confident that, since our current error graphics do not show the lobular aspect of Fig. 1(b), which means that the truncation of the multipolar expansion is an important error source, our results can be improved very easily even in the case of the strong gravitational field of a compact source of realistic mass.

Table 1 Some CMMR values and relative error with respect to AKM in $(c = G = B = 1)$ units. In other units, the three models rotate at $\nu = 102\,\mathrm{Hz}$ and their masses in $M_\odot$ appear in the first row.

$M_\odot$	$M_0 = 0.054,\ M_b = 0.06$		$M_0 = 0.51,\ M_b = 0.6$		$M_0 = 1.39,\ M_b = 1.7$	
	CMMR	error	CMMR	error	CMMR	error
ω	0.24		0.24		0.24	
M_b	7.9997e-4	3.2e-5	7.95e-3	6.1e-3	0.02241	0.0406
M_0	7.104e-4		6.772e-3		0.0184	
J_1	8.3609e-8	1e-4	3.586e-6	0.012	1.82e-5	0.034
R_{circ}	0.034769	3.6e-5	0.07287	8.6e-3	0.102	0.068
E_{bind}	8.8730e-6	2.8e-4	3.766e-4	0.040	1.75e-3	0.19
z_p	0.021133	7.6e-5	0.1079	0.016	0.24	0.13
V_0	-0.020913	4.6e-5	-0.103	0.010	-0.221	0.087
r_p	0.033914	4.7e-5	0.0657	9.8e-3	0.0831	0.09
r_e	0.034055	3.8e-5	0.06593	9.8e-3	0.0834	0.091
r_{ratio}	0.995862	9.4e-6	0.9964	2.4e-5	0.997	3e-4
ε_c	4.5935	6.8e-5	4.712	0.019	5.56	0.21
p_c	0.043679	2.4-4	0.254	0.052	0.631	0.37
h_c	0.908786	1.7e-6	0.9522	2e-3	1.025	0.042

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